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Interactions along Brownian paths in $\mathbb{R}^d, d \leqslant 5$

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Abstract
Via the Posilicano method, Schrödinger operators with singular potentials supported by Brownian paths in the configuration space $\mathbb{R}^d$, $1 \leqslant d \leqslant 5$, are constructed. The essential, absolutely continuous and singular continuous spectra are determined almost surely (with respect to Wiener measure). It is shown that the set of positive eigenvalues is discrete and that the wave operators exist and are asymptotically complete almost surely; if $d \geqslant 3$ then the set of positive eigenvalues is even empty almost surely. A trace formula for the number (counting multiplicities) of negative eigenvalues is derived.

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1. Introduction

In a wide variety of models in quantum field theory, one studies a family (H_ω) of Schrödinger operators in $L^2(\mathbb{R}^4, \lambda^4)(\lambda^d$ being the Lebesgue measure) with potentials supported by a Brownian path. Here severe mathematical problems arise from the very beginning. Due to the fact that the c_1-capacity of a 'typical path of a Brownian particle in $\mathbb{R}^4$' equals zero, Kato's quadratic form method cannot be used in order to define the operator H_ω (cf the introduction in [Bra] for a detailed discussion of this point).

Instead one has worked with ultraviolet cutoff [Cher] or nonstandard analysis [AFHL]. The spectral analysis of the operators constructed via these methods is, however, extremely difficult; in fact, virtually nothing is known about their spectra.

Recently, Posilicano [Pos1] presented a new method for the construction of singularly perturbed self-adjoint operators (cf also [Bra, Pos2]). In particular, he demonstrated that his method can be used for the construction of Schrödinger operators with potentials supported by Brownian paths if the dimension d of $\mathbb{R}^d$ is less than or equal to 5 [Pos1, example 3.6]. In this paper, we shall provide a detailed spectral analysis of a particular class of such operators; our operators are chosen such that their resolvents have a fairly simple form. We shall determine the essential spectra, prove existence and completeness of wave operators, absence of singular continuous spectra and positive eigenvalues, and derive a trace formula for the expectation value of the number (counting multiplicities) of negative eigenvalues.

Our operators are, in particular, self-adjoint extensions of the restriction S_ω^T of the free Hamiltonian to the space of smooth functions with compact support in the complement of the Brownian path Γ_ω^T (defined by (2.6) below). For almost all (w.r.t. to the Wiener measure) ω the symmetric operator S_ω^T is lower semibounded and has infinite deficiency indices. It has been shown that the spectra of self-adjoint extensions of lower semibounded symmetric operators with infinite deficiency indices strongly depend on the special choice of the self-adjoint extension (cf, e.g., [ABN]). In particular, for other classes of Schrödinger operators with potentials supported by Brownian paths (cf [Pos1]) one may get spectral properties different from those derived for the special class discussed in this paper.

Finally, let us mention an important open problem. The operators discussed in this paper are not form-local in the sense of the definition given in [Sh2]. It is an interesting and difficult open problem to obtain lower semibounded, form-local singular perturbations supported on Brownian paths. For the solution of this problem, it may be necessary to choose the self-adjoint extensions of S_ω^T in a more complicated way. This is due to the strong irregularity of the Brownian paths, as a simpler situation described in [Sh1] suggests.

2. Preliminaries and notation

2.1. Capacity and quasi-continuity

$L^2(\mathbb{R}^d) = L^2(\mathbb{R}^d, \lambda^d)$ denotes the space of (equivalence classes) of functions which are square-integrable w.r.t. the Lebesgue measure λ^d and $\hat{f}$ the Fourier transform of f. Let $s > 0. H^s(\mathbb{R}^d)$ denotes the Sobolev space of all $f \in L^2(\mathbb{R}^d)$ such that

$$\|f\|_{H^s} := \left(\int (1+x^2)^{s/2} |\hat{f}(x)|^2 \lambda^d(\mathrm{d}x)\right)^{1/2} < \infty. \tag{2.1}$$

The c_s-capacity of the compact set $K \subset \mathbb{R}^d$ is defined by

$$c_s(K) := \inf \|f\|_{H^s}^2,$$

where the infimum is taken over all f in the space $C_0^\infty(\mathbb{R}^d)$ of smooth functions f with compact support satisfying $f(x) \geqslant 1$ for all $x \in K$. The c_s-capacity of an arbitrary Borel set B is defined by

$$c_s(B) := \sup c_s(K), \tag{2.2}$$

where the supremum is taken over all compact subsets of B.

The function $g : \mathbb{R}^d \longrightarrow \mathbb{C}$ is quasi-continuous w.r.t. the c_s-capacity if and only if for every $\varepsilon > 0$ there exists an open subset O_ε of $\mathbb{R}^d$ such that

$$c_s(O_\varepsilon) < \varepsilon$$

and the restriction of g to the complement $\mathbb{R}^d \backslash O_\varepsilon$ is continuous. Every $f \in H^s(\mathbb{R}^d)$ has a representative $\tilde{f}$ which is quasi-continuous w.r.t. the c_s-capacity. If $\tilde{f}$ and f° are representatives of $f \in H^s(\mathbb{R}^d)$ and quasi-continuous w.r.t. the c_s-capacity then the c_s-capacity of the set $\{x \in \mathbb{R}^d : \tilde{f}(x) \neq f^\circ(x)\}$ equals zero. In the present paper $\tilde{f}$ denotes any representative of $f \in H^s(\mathbb{R}^d)$ which is quasi-continuous w.r.t. the c_s-capacity; this notation does not indicate which s is meant, but this will always be clear from the context.

If $\mu(B) = 0$ for every Borel set B satisfying $c_s(B) = 0$ and

$$\int |\tilde{f}|^2 \,\mathrm{d}\mu < \infty, \qquad f \in H^s(\mathbb{R}^d), \tag{2.3}$$

then we can define the mapping $J_{s\mu} : H^s(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d, \mu)$ by

$$J_{s\mu} f := \tilde{f} \qquad \mu\text{-a.e.}, \qquad f \in H^s(\mathbb{R}^d). \tag{2.4}$$

2.2. *Wiener measure and occupation time measure*

Ω denotes the space $C(\mathbb{R}_+, \mathbb{R}^d)$ of continuous functions $\omega : \mathbb{R}_+ = [0, \infty) \longrightarrow \mathbb{R}^d$ and $\mathbb{W}$ the Wiener measure on Ω. With $0 < T < \infty$ being fixed, the occupation time measure μ_ω^T on the Borel algebra $\mathcal{B}(\mathbb{R}^d)$ of $\mathbb{R}^d$ is defined via

$$\mu_\omega^T(B) := \lambda^1(\{t : 0 \leqslant t \leqslant T, \omega(t) \in B\}), \qquad B \in \mathcal{B}(\mathbb{R}^d). \tag{2.5}$$

The topological support of the occupation time measure μ_ω^T equals the set

$$\Gamma_\omega^T := \{\omega(t) : 0 \leqslant t \leqslant T\}. \tag{2.6}$$

2.3. *Singular perturbations*

Let $s, \alpha > 0$. We put

$$G_{s\alpha} := (-\Delta + \alpha)^{-s} \qquad \text{and} \qquad G_\alpha := G_{1\alpha} = (-\Delta + \alpha)^{-1} \tag{2.7}$$

where $-\Delta$ is the self-adjoint operator in $L^2(\mathbb{R}^d)$ defined by

$$D(-\Delta) := H^2(\mathbb{R}^d), \qquad -\Delta f := -\sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}, \qquad f \in H^2(\mathbb{R}^d),$$

and derivatives have to be understood in the distributional sense.

An operator H belongs to the set $\mathcal{A}_\omega^T$ if and only if

$$\begin{aligned} &H \text{ is a self-adjoint operator in } L^2(\mathbb{R}^d),\\ &C_0^\infty\big(\mathbb{R}^d \setminus \Gamma_\omega^T\big) \subset D(H),\\ &Hf = -\Delta f, \qquad f \in C_0^\infty\big(\mathbb{R}^d \setminus \Gamma_\omega^T\big),\\ &H \neq -\Delta. \end{aligned} \tag{2.8}$$

If there exists an $s < 2$ such that $\mu_\omega^T(B) = 0$ for every Borel set B satisfying $c_s(B) = 0$ and

$$\mu_\omega^T(B) = 0, \qquad \text{if} \quad c_s(B) = 0, \quad \text{and} \quad \int |\tilde{f}|^2 \, d\mu_\omega^T < \infty, \qquad f \in H^s(\mathbb{R}^d), \tag{2.9}$$

then we can define the mapping $J_\omega^T : H^2(\mathbb{R}^d) \longrightarrow L^2\big(\mathbb{R}^d, \mu_\omega^T\big)$ by

$$J_\omega^T f := \tilde{f} \qquad \mu_\omega^T\text{-a.e.}, \qquad f \in H^2(\mathbb{R}^d) \qquad \big(\text{i.e. } J_\omega^T = J_{2\mu_\omega^T}\big) \tag{2.10}$$

and there exists a unique operator $H_{\omega\alpha}^T \in \mathcal{A}_\omega^T$ such that $-\alpha$ belongs to the resolvent set of $H_{\omega\alpha}^T$ and

$$\big(H_{\omega\alpha}^T + \alpha\big)^{-1} = G_\alpha + \big(J_\omega^T G_\alpha\big)^* \big(J_\omega^T G_\alpha\big), \tag{2.11}$$

cf [Bra, theorem 9].

2.4. Wave operators and Schatten classes

The wave operators $W^{\pm}(H, -\Delta)$ exist provided

$$W^{\pm}(H, -\Delta) f := \lim_{t \mp \infty} \mathrm{e}^{\mathrm{i}tH} \mathrm{e}^{\mathrm{i}t\Delta} f$$

exist for every $f \in L^2(\mathbb{R}^d)$. The wave operators $W^{\pm}(H, -\Delta)$ are asymptotically complete if and only if

$$\operatorname{ran}(W^{+}(H, -\Delta)) = \operatorname{ran}(W^{-}(H, -\Delta)) = (\mathcal{H}^{pp}(H))^{\perp}$$

(i.e. every state f can be decomposed into the orthogonal sum of a bound state f_b and a state f_s such that the system behaves asymptotically as a free system provided the initial state equals f_s). The wave operators $W^{\pm}(H, -\Delta)$ are asymptotically complete if and only if the singular continuous spectrum $\sigma_{\mathrm{sc}}(H)$ is empty and the operators $W^{\pm}(H, -\Delta)$ are complete, i.e,

$$\operatorname{ran}(W^{+}(H, -\Delta)) = \operatorname{ran}(W^{-}(H, -\Delta)) = \mathcal{H}^{ac}(H).$$

Here $\mathcal{H}^{pp}(H)$ and $\mathcal{H}^{ac}(H)$ denote the pure point spectral subspace (the closure of the span of the eigenvectors) of H and the absolutely continuous spectral subspace of H, respectively.

In order to prove existence and completeness of wave operators one often uses Schatten classes. Let $C : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a compact linear bounded mapping. There exists an orthonormal basis $\{e_i\}_{i \in I}$ of $\mathcal{H}_2$ and non-negative numbers λ_i, $i \in I$, such that

$$\sqrt{CC^*} e_i = \lambda_i e_i, \qquad i \in I.$$

The family $\{\lambda_i\}_{i \in I}$ is unique up to permutations. We put

$$\|C\|_{\mathbb{S}_p} := \left(\sum_{i \in I} \lambda_i^p \right)^{1/p} \qquad (\leqslant \infty), \qquad 0 < p < \infty.$$

C belongs to the Schatten class of order p provided $\|C\|_{\mathbb{S}_p} < \infty$. We define $\|C\|_{\mathbb{S}_p} = \infty$ if C is not compact.

We shall repeatedly use the following well-known facts: along with C also $B_1 C B_2$ and the adjoint C^* belong to the Schatten class $\mathbb{S}_p$ for all bounded operators B_1 and B_2. Moreover, $CK \in \mathbb{S}_r$ provided $C \in \mathbb{S}_p$, $K \in \mathbb{S}_q$ and $1/p + 1/q = 1/r$.

3. Compactness and Schatten norms

Let $d \leqslant 5$. Our first goal is to show that the condition (2.9) is satisfied for $\mathbb{W}$-a.a. $\omega \in \Omega$. As mentioned, this guarantees that for $\mathbb{W}$-a.a. $\omega \in \Omega$ there exists a unique operator $H^T_{\omega\alpha} \in \mathcal{A}^T_\omega$ such that $-\alpha$ belongs to the resolvent set of $H^T_{\omega\alpha}$ and (2.11) holds. If the dimension d is larger than 5 then the c_2-capacity of the set Γ^T_ω (cf (2.1), (2.2) and (2.6)) equals zero $\mathbb{W}$-a.s. and the set $\mathcal{A}^T_\omega$ is empty $\mathbb{W}$-a.s.

We shall prove (2.3) with the aid of lemma 3.1 below which might be useful in other contexts, too. Let $s, \alpha > 0$ and $d \in \mathbb{N}$.

There exist rotationally symmetric functions $k_{s\alpha} : \mathbb{R}^d \longrightarrow [0, \infty]$ and $g_{s\alpha} : \mathbb{R}^d \longrightarrow [0, \infty]$ satisfying

$$\hat{k}_{s\alpha}(p) = (p^2 + \alpha)^{-s/2}, \qquad \hat{g}_{s\alpha}(p) = (p^2 + \alpha)^{-s}, \qquad \lambda^d\text{-a.e.}, \tag{3.1}$$

cf [SW]. We choose $k_{s\alpha}$ and $g_{s\alpha}$ such that they are continuous on $\mathbb{R}^d$ if possible (i.e. if $s > d$ respectively $s > d/2$); otherwise we choose them such that they are continuous on $\mathbb{R}^d \backslash \{0\}$ and equal to ∞ at 0. $g_{s\alpha}$ is the convolution kernel of the operator $(-\Delta + \alpha)^{-s}$ on $L^2(\mathbb{R}^d, \lambda^d)$.

Lemma 3.1. *Let $G^{\mu}_{s\alpha}$ be the integral operator with kernel $g_{s\alpha}(x-y)$ (cf (3.1)) in $L^2(\mathbb{R}^d, \mu)$. If $G^{\mu}_{s\alpha}$ is bounded then the measure μ does not charge any set with c_s-capacity zero and*

$$\int |\tilde{v}|^2 \,\mathrm{d}\mu \leqslant \left\|G^{\mu}_{s\alpha}\right\| ((-\Delta+\alpha)^{s/2} v, (-\Delta+\alpha)^{s/2} v)_{L^2(\mathbb{R}^d,\lambda^d)}, \qquad v \in H^s(\mathbb{R}^d). \tag{3.2}$$

The estimate (3.2) is sharp.

Proof. Denote by $K^{\mu}_{s\alpha}$ the integral operator with kernel $k_{s\alpha}(x-y)$ (cf (3.1)) from $L^2(\mathbb{R}^d, \mu)$ to $L^2(\mathbb{R}^d, \lambda^d)$. Then the adjoint operator $K^{\mu*}_{s\alpha}$ is the integral operator from $L^2(\mathbb{R}^d, \lambda^d)$ to $L^2(\mathbb{R}^d, \mu)$ with the same kernel $k_{s\alpha}(x-y)$.

Let $f \in L^2(\mathbb{R}^d, \mu)$, $f \geqslant 0\mu$-a.e. Then

$$\begin{aligned}
\iint k_{s\alpha}(x-y) f(y)\mu(\mathrm{d}y) \int k_{s\alpha}(x-z) f(z)\mu(\mathrm{d}z)\lambda^d(\mathrm{d}x) \\
&= \iiint k_{s\alpha}(x-y)k_{s\alpha}(x-z)\lambda^d(\mathrm{d}x) f(y)\mu(\mathrm{d}y) f(z)\mu(\mathrm{d}z) \\
&= \int f(y) \int g_{s\alpha}(y-z) f(z)\mu(dz)\mu(dy) \\
&= \left(f, G^{\mu}_{s\alpha} f\right)_{L^2(\mathbb{R}^d,\mu)} \\
&\leqslant \left\|G^{\mu}_{s\alpha}\right\| \|f\|^2_{L^2(\mathbb{R}^d,\mu)} < \infty.
\end{aligned}$$

In the second step we have used that

$$\int k_{s\alpha}(x-y)k_{s\alpha}(x-z)\lambda^d(\mathrm{d}x) = g_{s\alpha}(y-z).$$

Thus we arrive at

$$\left\|K^{\mu*}_{s\alpha}\right\|^2 \leqslant \left\|G^{\mu}_{s\alpha}\right\| < \infty.$$

For every f in the Schwartz space of rapidly decreasing smooth functions the function

$$v(\cdot) := \int k_{s\alpha}(\cdot - y) f(y)\lambda^d(\mathrm{d}y)$$

also belongs to Schwartz space $\mathcal{S}(\mathbb{R}^d)$; in particular, v is continuous. Note that v is a representative of both $(-\Delta+\alpha)^{-s/2} f$ and $K^{\mu*}_{s\alpha} f$. Moreover,

$$\begin{aligned}
\int |v|^2 \,\mathrm{d}\mu &= \left\|K^{\mu*}_{s\alpha} f\right\|^2_{L^2(\mathbb{R}^d,\mu)} \leqslant \left\|K^{\mu*}_{s\alpha}\right\|^2 \|f\|^2_{L^2(\mathbb{R}^d,\lambda^d)} \\
&\leqslant \left\|G^{\mu}_{s\alpha}\right\| (v, (-\Delta+\alpha)^s v)_{L^2(\mathbb{R}^d,\lambda^d)} \leqslant c\|v\|^2_{H^s(\mathbb{R}^d)}
\end{aligned} \tag{3.3}$$

for some finite constant c independent of v.

If the c_s-capacity $c_s(K)$ of the compact set K equals zero then there exist v_n in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ satisfying

$$v_n \geqslant 1 \text{ on } K \qquad \text{and} \qquad \|v_n\|_{H^s(\mathbb{R}^d)} \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

By (3.3), it follows that $c_s(K) = 0$. By the inner regularity of the c_s-capacity and the measure μ, this implies that $c_s(B) = 0$ for every Borel set B such that $\mu(B) = 0$.

Let $v \in H^s(\mathbb{R}^d)$. Take any v_n, $n \in \mathbb{N}$, in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ converging to v in $H^s(\mathbb{R}^d)$ as n tends to infinity. By (3.3), there exists an $h \in L^2(\mathbb{R}^d, \mu)$ such that

$$v_n \longrightarrow h \qquad \text{as} \qquad n \longrightarrow \infty \qquad \text{in } L^2(\mathbb{R}^d, \mu).$$

Moreover, there exists a subsequence $\{v_{n_j}\}$ of $\{v_n\}$ such that

$$v_{n_j} \longrightarrow \tilde{v} \qquad c_s\text{-q.e.} \qquad \text{as } n \longrightarrow \infty.$$

Since the measure μ does not charge any set with c_s-capacity zero it follows that $\tilde{v} = h$ almost everywhere with respect to the measure μ, and that $\tilde{v} \in L^2(\mathbb{R}^d, \mu)$ and

$$\begin{aligned}\int |\tilde{v}|^2 \,\mathrm{d}\mu &= \lim_{n\longrightarrow\infty} \int |v_n|^2 \,\mathrm{d}\mu \\ &\leqslant \lim_{n\longrightarrow\infty} \left\|G^{\mu}_{s\alpha}\right\| (v_n, (-\Delta+\alpha)^s v_n)_{L^2(\mathbb{R}^d,\lambda^d)} \\ &= \left\|G^{\mu}_{s\alpha}\right\| ((-\Delta+\alpha)^{s/2} v, (-\Delta+\alpha)^{s/2} v)_{L^2(\mathbb{R}^d,\lambda^d)}.\end{aligned}$$

Since $G^{\mu}_{s\alpha} = J_{s\mu}(J_{s\mu}G_{s\alpha})^* = \left(J_{s\mu}G^{1/2}_{s\alpha}\right)\left(J_{s\mu}G^{1/2}_{s\alpha}\right)^* = K^{\mu *}_{s\alpha} K^{\mu}_{s\alpha}$ the operator $G^{\mu}_{s\alpha}$ is non-negative and self-adjoint, and

$$\left\|G^{\mu}_{s\alpha}\right\| = \left\|K^{\mu}_{s\alpha}\right\|^2. \tag{3.4}$$

We choose a sequence $\{f_n\}$ in $\mathcal{S}(\mathbb{R}^d)$ such that $\|f_n\|_{L^2(\mathbb{R}^d,\lambda^d)} = 1$ for every $n \in \mathbb{N}$ and $\left\|K^{\mu *}_{s\alpha} f_n\right\|_{L^2(\mathbb{R}^d,\lambda^d)} \longrightarrow \left\|K^{\mu *}_{s\alpha}\right\|$. We put $v_n := (-\Delta+\alpha)^{-s/2} f_n, n \in \mathbb{N}$. Then $(v_n, (-\Delta+\alpha)^s v_n) = 1$ for every $n \in \mathbb{N}$ and (3.3) and (3.4) yield

$$\int |v_n|^2 \,\mathrm{d}\mu \longrightarrow \left\|G^{\mu}_{s\alpha}\right\|, \qquad \text{as} \qquad n \longrightarrow \infty,$$

i.e, inequality (3.2) is sharp. □

Remark 3.2. (a) With the aid of the above lemma we can immediately rediscover a well-known result on measures in Kato classes. Let $0 < s < d/2$. Let μ be a measure in the Kato class w.r.t. the operator $(-\Delta)^s$, i.e,

$$\lim_{\varepsilon\longrightarrow 0} \sup_{x\in\mathbb{R}^d} \int_{|y-x|\leqslant\varepsilon} \frac{1}{|x-y|^{d-2s}} \mu(\mathrm{d}y) = 0.$$

Then the Schur test in combination with the facts that $g_{s\alpha}(x)$ tends to zero uniformly on $\{x : |x| > \varepsilon\}$ as α tends to infinity and that there exists a finite constant c independent of α such that $g_{s\alpha}(x) \leqslant c|x|^{2s-d}$ for all $x \in \mathbb{R}^d$ yields that the operator norm $\left\|G^{\mu}_{s\alpha}\right\|$ of $g^{\mu}_{s\alpha}$ tends to zero as α tends to infinity. Thus (3.2) implies that $\int |\tilde{v}|^2 \,\mathrm{d}\mu$ is an infinitesimal small form perturbation of the operator $(-\Delta)^s$. (b) For $s \leqslant 1$ the operator $(-\Delta)^s$ is associated with a Dirichlet form; we refer to [Amor] for a partial generalization of the above lemma in the Dirichlet case.

By lemma 3.1, (2.3) holds provided that the operator $G^{\mu}_{s\alpha}$ is bounded for some $s < 2$. Actually, this operator even belongs to the Schatten class of order 4 if $s > d/2 - 1$:

Lemma 3.3. *Let $s > d/2 - 1, \alpha > 0$ and for $\omega \in \Omega$ let $G^{\mu^T_\omega}_{s\alpha}$ be the integral operator in $L^2\left(\mathbb{R}^d, \mu^T_\omega\right)$ with the kernel $g_{s\alpha}(x-y)$ defined by (3.1). Then*

$$\mathbb{E}\left\|G^{\mu^T_\cdot}_{s\alpha}\right\|^4_{\mathbb{S}_4} < \infty,$$

i.e., the expectation value (w.r.t. to the Wiener measure $\mathbb{W}$) of $\left\|G^{\mu^T_\cdot}_{s\alpha}\right\|^4_{\mathbb{S}_4}$ is finite. In particular, $\mathbb{W}$-a.s. the operator $G^{\mu^T_\omega}_{s\alpha}$ belongs to the Schatten class $\mathbb{S}_4$ of order 4. Moreover,

$$\mathbb{E}\left\|G^{\mu^T_\cdot}_{s\alpha}\right\|^4_{\mathbb{S}_4} \longrightarrow 0, \qquad \alpha \longrightarrow \infty.$$

Proof. First let μ be any positive Radon measure on $\mathbb{R}^d$. Then

$$\begin{aligned}\|G^{\mu}_{s\alpha}\|^4_{\mathbb{S}_4} &= \|(G^{\mu}_{s\alpha})^2\|^2_{\mathbb{S}_2}\\ &= \iint \left(\int g_{s\alpha}(x-z)g_{s\alpha}(z-y)\mu(\mathrm{d}z)\right)^2 \mu(\mathrm{d}x)\mu(\mathrm{d}y)\\ &= \iiiint g_{s\alpha}(x-z)g_{s\alpha}(z-y)g_{s\alpha}(x-a)g_{s\alpha}(a-y)\mu(\mathrm{d}z)\mu(\mathrm{d}x)\mu(\mathrm{d}y)\mu(\mathrm{d}a).\end{aligned}$$

If μ equals the occupation time measure μ^T_ω then this implies, by the general transformation theorem, that

$$\begin{aligned}\|G^{\mu^T_\omega}_{s\alpha}\|^4_{\mathbb{S}_4} = \int_0^T\int_0^T\int_0^T\int_0^T & g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4))\\ &\times g_{s\alpha}(\omega(t_4)-\omega(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4.\end{aligned} \tag{3.5}$$

For every element π of the symmetric group S_4 let

$$M_\pi := \{(t_1,t_2,t_3,t_4) \in [0,T]^4 : t_{\pi(1)} < t_{\pi(2)} < t_{\pi(3)} < t_{\pi(4)}\}.$$

Up to a set with Lebesgue measure zero the domain of integration in (3.5), i.e. the set $[0,T]^4$, equals the disjoint union of the 24 sets M_π, $\pi \in S_4$.

By using Gaussian kernels

$$p_t(x) := (2\pi|t|)^{-d/2}\,\mathrm{e}^{-\frac{x^2}{2|t|}}$$

we can derive an expression for the expectation value of the integral over the set M_π for every $\pi \in S_4$. For instance, in the case $\pi(j) = j$ for $j = 1,2,3,4$ we get

$$\begin{aligned}\mathbb{E}\int_{0\leqslant t_1<t_2<t_3<t_4\leqslant T} & g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4))\\ &\times g_{s\alpha}(\omega(t_4)-\omega(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4\\ &= \int_{0\leqslant t_1<t_2<t_3<t_4\leqslant T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} p_{t_2-t_1}(x)p_{t_3-t_2}(y)p_{t_4-t_3}(z)g_{s\alpha}(x+y)g_{s\alpha}(y)\\ &\times g_{s\alpha}(x+y+z)g_{s\alpha}(y+z)\lambda^d(x)\lambda^d(y)\lambda^d(z)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4.\end{aligned} \tag{3.6}$$

The function $g_{s\alpha}$ tends exponentially fast to zero at infinity. Moreover, it is bounded if $2s > d$, has a logarithmic singularity at 0, if $2s = d$, and satisfies

$$g_{s\alpha}(x) \leqslant c_\alpha |x|^{2s-d}, \qquad x \in \mathbb{R}^d, \tag{3.7}$$

for some finite constant c_α if $2s < d$. $\limsup_{\alpha\longrightarrow\infty} c_\alpha < \infty$. Moreover, $g_{s\alpha}(x) \longrightarrow 0$, as $\alpha \longrightarrow \infty$, for every $x \in \mathbb{R}^d\backslash\{0\}$. In what follows, we shall treat the last case, $2s < d$; the other two cases can be treated in an analogous way and are even more simple.

By (3.7), the integrand on the right-hand side of (3.6) is, up to a constant, bounded by

$$p_{t_2-t_1}(x)p_{t_3-t_2}(y)p_{t_4-t_3}(z)|x+y|^{2s-d}|y|^{2s-d}|x+y+z|^{2s-d}|y+z|^{2s-d}.$$

A straightforward but tedious computation shows that

$$\begin{aligned}\int_{0\leqslant t_1<t_2<t_3<t_4\leqslant T}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} & p_{t_2-t_1}(x)p_{t_3-t_2}(y)p_{t_4-t_3}(z)|x+y|^{2s-d}|y|^{2s-d}\\ &\times |x+y+z|^{2s-d}|y+z|^{2s-d}\lambda^d(x)\lambda^d(y)\lambda^d(z)\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4 < \infty\end{aligned} \tag{3.8}$$

provided $s > d/2 - 1$. Thus

$$\begin{aligned}\mathbb{E}\int_{0\leqslant t_1<t_2<t_3<t_4\leqslant T} & g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4))\\ &\times g_{s\alpha}(\omega(t_4)-\omega(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4 < \infty\end{aligned}$$

for every $\alpha > 0$ and

$$\mathbb{E}\int_{0\leqslant t_1<t_2<t_3<t_4\leqslant T} g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4)) \times g_{s\alpha}(\omega(t_4)-\omega(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4 \longrightarrow 0, \qquad \alpha\longrightarrow\infty.$$

The remaining 23 domains of integration can be treated in a similar manner and we get that

$$\mathbb{E}\int_{[0,T]^4} g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4)) \times g_{s\alpha}(\omega(t_4)-\omega(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4 < \infty$$

for every $\alpha > 0$ and

$$\mathbb{E}\int_{[0,T]^4} g_{s\alpha}(\omega(t_1)-\omega(t_3))g_{s\alpha}(\omega(t_3)-\omega(t_2))g_{s\alpha}(\omega(t_1)-\omega(t_4)) \times g_{s\alpha}(\omega(t_4)-\omega(t_2))\,\mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{d}t_3\,\mathrm{d}t_4 \longrightarrow 0, \qquad \alpha\longrightarrow\infty.$$

By (3.5), we have proved the lemma. □

4. Wave operators, continuous spectral subspaces and positive eigenvalues

In this section, we shall present results on the scattering theory for the operators $H^T_{\omega\alpha}$ and related results on their spectra. Moreover, we shall prove the absence of singular continuous spectra and, for $d \geqslant 3$, also the absence of positive eigenvalues.

Theorem 4.4. *Let the dimension d of $\mathbb{R}^d$ be less than 5 or equal to 5 and $\alpha > 0$. For $\mathbb{W}$-a.a. $\omega \in \Omega$ let $H^T_{\omega\alpha}$ be the self-adjoint operator defined by (2.7), (2.10) and (2.11). Then the following is true for $\mathbb{W}$-a.a. $\omega \in \Omega$:*

(i) The essential spectrum of $H^T_{\omega\alpha}$ equals $[0,\infty)$.
(ii) The wave operators $W^\pm\big(H^T_{\omega\alpha}, -\Delta\big)$ exist and are asymptotically complete.
(iii) The singular continuous spectrum of $H^T_{\omega\alpha}$ is empty, the set of the positive eigenvalues of $H^T_{\omega\alpha}$ is discrete and every positive eigenvalue of $H^T_{\omega\alpha}$ (if there is any) has finite multiplicity.
(iv) The absolutely continuous part of $H^T_{\omega\alpha}$ is unitarily equivalent to the operator $-\Delta$ and, in particular, the absolutely continuous spectrum of $H^T_{\omega\alpha}$ equals $[0,\infty)$.
(v) If $d \geqslant 3$ then the operator $H^T_{\omega\alpha}$ has no positive eigenvalue.

Proof. (i) We have

$$G^{\mu^T_\omega}_{2\alpha} = J^T_\omega G^{1/2}_{2\alpha}\big(J^T_\omega G^{1/2}_{2\alpha}\big)^* = J^T_\omega G_\alpha\big(J^T_\omega G_\alpha\big)^*. \tag{4.9}$$

Since $2 > d/2 - 1$, this equation and lemma 3 imply that

$$J^T_\omega G_\alpha \in \mathbb{S}_8 \qquad \text{for} \quad \mathbb{W}\text{-a.a. } \omega\in\Omega. \tag{4.10}$$

By (2.11) and (4.10),

$$\big(H^T_{\omega\alpha}+\alpha\big)^{-1} - (-\Delta+\alpha)^{-1} = J^T_\omega G_\alpha\big(J^T_\omega G_\alpha\big)^* \in \mathbb{S}_4 \qquad \text{for} \quad \mathbb{W}\text{-a.a. } \omega\in\Omega. \tag{4.11}$$

Since every operator in $\mathbb{S}_p$ is, in particular, compact, and the operators $H^T_{\omega\alpha}$ and $-\Delta$ are self-adjoint, Weyl's essential spectrum theorem together with (4.11) implies the assertion (i).

(ii) The wave operators exist and are asymptotically complete provided that the singular continuous spectra are empty and the wave operators exist and are complete. We shall prove the absence of singular continuous spectra below, under (iii).

The wave operators exist and are complete provided that there exists an $N \in \mathbb{N}$ such that the operator

$$D^T_{\alpha\omega N} := (H_{\omega\alpha} + \alpha)^{-N} - (-\Delta + \alpha)^{-N}$$

is compact and

$$(H_{\omega\alpha} + \alpha)^{-N} D^T_{\alpha\omega N} (-\Delta + \alpha)^{-N} \in \mathbb{S}_1, \tag{4.12}$$

cf [Dem].

It follows immediately from (4.11) and the identity

$$D^T_{\alpha\omega N} = \sum_{j=0}^{N-1} (H_{\omega\alpha} + \alpha)^{-j} ((H_{\omega\alpha} + \alpha)^{-1} - (-\Delta + \alpha)^{-1})(-\Delta + \alpha)^{-(N-1-j)}$$

that the operator $D^T_{\alpha\omega N}$ is compact for $\mathbb{W}$-a.a. $\omega \in \Omega$. Thus we need only to prove that (4.12) is true $\mathbb{W}$-a.s. for some $N \in \mathbb{N}$.

For $k > d/2$ the integral operator $J^T_\omega G^k_\alpha$ has a continuous convolution kernel vanishing exponentially fast at infinity. Thus

$$\begin{aligned} &\left(J^T_\omega G_\alpha\right)^* J^T_\omega G_\alpha G^j_\alpha \in \mathbb{S}_2, \qquad j > d/2 - 1, \\ &G^j_\alpha \left(J^T_\omega G_\alpha\right)^* J^T_\omega G_\alpha \in \mathbb{S}_2, \qquad j > d/2 - 1. \end{aligned} \tag{4.13}$$

Let $N \in \mathbb{N}$ and $N > d$. Since

$$\left(H^T_{\omega\alpha} + \alpha\right)^{-1} - (-\Delta + \alpha)^{-1} = \left(J^T_\omega G_\alpha\right)^* J^T_\omega G_\alpha$$

(cf (2.11)), the operator $D^T_{\alpha\omega N}$ is the sum of $2^N - 1$ terms where every term has the form

$$A\left(J^T_\omega G_\alpha\right)^* J^T_\omega G_\alpha G^j_\alpha B$$

or

$$A G^j_\alpha \left(J^T_\omega G_\alpha\right)^* J^T_\omega G_\alpha B$$

or

$$A\left(J^T_\omega G_\alpha\right)^* J^T_\omega G_\alpha B \left(J^T_\omega G_\alpha\right)^* J^T_\omega G_\alpha C$$

for some bounded operators A, B, C and some $j > d/2 - 1$. By (4.11) and (4.13), each of these terms belongs to the Hilbert–Schmidt class $\mathbb{S}_2$. Thus

$$D^T_{\alpha\omega N} \in \mathbb{S}_2 \qquad \text{for} \quad \mathbb{W}\text{-a.a. } \omega \in \Omega \text{ (if } N > d\text{)}. \tag{4.14}$$

We have

$$\left(H^T_{\omega\alpha} + \alpha\right)^{-N} D^T_{\alpha\omega N} (-\Delta + \alpha)^{-N} = D^T_{\alpha\omega N} D^T_{\alpha\omega N} + (-\Delta + \alpha)^{-N} D^T_{\alpha\omega N} (-\Delta + \alpha)^{-N}.$$

For $\mathbb{W}$-a.a. $\omega \in \Omega$ the first term on the right-hand side belongs to the trace class $\mathbb{S}_1$ since it is the product of two Hilbert–Schmidt operators. The second term is the sum of 2^{N-1} operators where every operator has the form

$$A G^N_\alpha \left(J^T_\omega G_\alpha\right)^* B J^T_\omega G_\alpha G^N_\alpha C$$

for some bounded operators A, B, C. Applying again (4.13) we get that each of these 2^{N-1} operators is the product of two Hilbert–Schmidt operators and therefore also an operator in the trace class. Thus (4.12) holds $\mathbb{W}$-a.s. for every $N > d$.

(iii) Let $D := \{z \in \mathbb{C} : \mathrm{Re}(z) > 0 \text{ or } \mathrm{Im}(z) > 0\}$, and $D_{\mathrm{ext}} := D \cup \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$. It is sufficient to prove that for $\mathbb{W}$-a.a. $\omega \in \Omega$ there exists a discrete set C such that for every $f \in C^\infty_0(\mathbb{R}^d)$ the mapping

$$z \mapsto \left(f, \left(H^T_{\omega\alpha} + z\right)^{-1} f\right) D \longrightarrow \mathbb{C}$$

has an analytic continuation on D_{ext}. C may depend on ω.

In fact, suppose that such a discrete set C and such an analytic extension exist. Let $f \in C_0^\infty(\mathbb{R}^d)$. Let $-\infty < a < b < 0$ be such that $[a, b] \cap C = \emptyset$. Then there exists an $\varepsilon > 0$ such that

$$\{x + \mathrm{i}y : a \leqslant x \leqslant b, 0 \leqslant y \leqslant \varepsilon\} \cap C = \emptyset.$$

Since continuous mappings are bounded on compact sets and the mapping $z \mapsto \left(f, \left(H_{\omega\alpha}^T + z\right)^{-1} f\right)$ has a continuation on a neighbourhood of the compact set $\{x + \mathrm{i}y : a \leqslant x \leqslant b, 0 \leqslant y \leqslant \varepsilon\}$ we get

$$\sup_{a \leqslant x \leqslant b, 0 < y \leqslant \varepsilon} \left|\left(f, \left(H_{\omega\alpha}^T + z\right)^{-1} f\right)\right| < \infty.$$

Since the space $C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$ and by the limiting absorption principle ([RS4], theorem XIII.19), this implies that

$$\sigma_{\mathrm{sc}}\left(-H_{\omega\alpha}^T\right) \cap (a, b) = \emptyset = \sigma_p\left(-H_{\omega\alpha}^T\right) \cap (a, b).$$

Since C is discrete it follows that $\sigma_{\mathrm{sc}}\left(-H_{\omega\alpha}^T\right) \cap (-\infty, 0]$ is at most countable. This is only possible if $\sigma_{\mathrm{sc}}\left(-H_{\omega\alpha}^T\right) \cap (-\infty, 0) = \emptyset$. Moreover, by (i) and the fact that $\sigma_{\mathrm{sc}}\left(-H_{\omega\alpha}^T\right) \subset \sigma_{\mathrm{ess}}\left(-H_{\omega\alpha}^T\right)$, we also have $\sigma_{\mathrm{sc}}\left(-H_{\omega\alpha}^T\right) \cap (0, \infty) = \emptyset$.

It remains to prove the existence of the mentioned continuation. By (4.10), $J_\omega^T G_\alpha$ is compact $\mathbb{W}$-a.s. and therefore J_ω^T is also compact $\mathbb{W}$-a.s. Trivially, the range of J_ω^T is dense in $L^2\left(\mathbb{R}^d, \mu_\omega^T\right)$. Thus, by [Bra, theorem 3], $(-\infty, -\alpha]$ belongs to the resolvent set of $H_{\omega\alpha}^T$ and

$$\left(H_{\omega\alpha}^T + \beta\right)^{-1} = G_\beta + \left(J_\omega^T G_\beta\right)^* \left(I - (\alpha - \beta) J_\omega^T G_\alpha \left(J_\omega^T G_\beta\right)^*\right)^{-1} J_\omega^T G_\beta, \qquad \beta > \alpha, \qquad \mathbb{W}\text{-a.s.} \tag{4.15}$$

In what follows let ω be any element of Ω such that J_ω^T is compact. Let

$$g_z(x) := \frac{1}{2\sqrt{z}} \mathrm{e}^{-\sqrt{z}|x|}, \qquad x \in \mathbb{R}, \quad d = 1,$$

respectively,

$$\frac{1}{(2\pi)^{d/2}} \left(\frac{|x|}{-\sqrt{-z}}\right)^{1-d/2} K_{d/2-1}(-\sqrt{z}|x|), \qquad x \in \mathbb{R}^d \backslash \{0\}, \qquad d > 1.$$

Here we choose the root as follows: $\sqrt{r \exp(\mathrm{i}\phi)} = \sqrt{r} \exp(\mathrm{i}\phi/2)$ for $r > 0$ and $-\pi/2 < \phi < 3\pi/2$. Then

$$\hat{g}_z(p) = \frac{1}{p^2 + z}, \qquad \mathrm{Re}(z) > 0 \text{ or } \mathrm{Im}(z) > 0,$$

(cf [SW]) and this definition of $g_z(x)$ is in accordance with (3.1). If $\mathrm{Re}(z) < 0$ and $\mathrm{Im}(z) \leqslant 0$ then g_z is not square-integrable w.r.t. the Lebesgue measure. Note that the function $z \mapsto g_z(x)$ is analytic on D_{ext} for $x \neq 0$ (every x if $d = 1$).

For $z \in D$ we define the operator G_z in $L^2(\mathbb{R}^d)$ by $G_z := (-\Delta + z)^{-1}$. For $z \in D_{\mathrm{ext}}$ let $G_z^{\mu_\omega^T}$ be the integral operator in $L^2\left(\mathbb{R}^d, \mu_\omega^T\right)$ with the kernel $g_z(x - y)$. By the preceding considerations, we need only to prove that there exists a discrete set C such that

$(\alpha) I - (\alpha - z) G_z^{\mu_\omega^T}$ is invertible in $L^2\left(\mathbb{R}^d, \mu_\omega^T\right)$ for every $z \in D_{\mathrm{ext}}$, and (β) the mapping $z \mapsto \left(f, G_z f + \left(J_\omega^T G_{\bar{z}}\right)^* \left[I - (\alpha - z) G_z^{\mu_\omega^T}\right]^{-1} J G_z f\right)$ is analytic on $D_{\mathrm{ext}} \backslash C$ for every $f \in C_0^\infty(\mathbb{R}^d)$.

A straightforward computation yields analyticity of the mapping $z \mapsto G_z^{\mu_\omega^T}$ and (α) and (β) follow from Fredholm's analytic theorem.

(iv) It is well known that the spectrum $\sigma(-\Delta)$ of $-\Delta$ equals $[0, \infty)$ and that $-\Delta$ equals its absolutely continuous part $(-\Delta)^{\mathrm{ac}}$. Since the wave operators $W^{\pm}(H^T_{\omega\alpha}, -\Delta)$ exist and are complete for $\mathbb{W}$-a.a. $\omega \in \Omega$ this implies, by [RS3, XI 3, proposition 1], that the wave operators $W^{\pm}(H^T_{\omega\alpha}, -\Delta)$ are unitary mappings from $L^2(\mathbb{R}^d, \lambda^d)$ onto the absolutely continuous spectral subspaces of $W^{\pm}(H^T_{\omega\alpha}, -\Delta)$ and

$$H^T_{\omega\alpha} = W^{\pm}(H^T_{\omega\alpha}, -\Delta)^{-1}(-\Delta)W^{\pm}(H^T_{\omega\alpha}, -\Delta), \qquad \mathbb{W}\text{-a.s}$$

In particular, the operators $H^T_{\omega\alpha}$ and $-\Delta$ have the same absolutely continuous spectrum and therefore

$$\sigma_{\mathrm{ac}}(H^T_{\omega\alpha}) = \sigma_{\mathrm{ac}}(-\Delta) = \sigma(-\Delta) = [0, \infty).$$

□

By the last theorem, the set of positive eigenvalues of the operator $H^T_{\omega\alpha}$ is discrete. In the case when $d \geqslant 3$ the complement of a typical path Γ^T_ω of a Brownian particle in $\mathbb{R}^d$ is connected. Together with a unique continuation theorem this provides a much stronger statement about positive eigenvalues in the case $d \geqslant 3$:

Theorem 4.5 Let $d \geqslant 3$. *For every $\omega \in \Omega$ let H^T_ω be any self-adjoint operator in $L^2(\mathbb{R}^d, \lambda^d)$ such that the space $C_0^\infty(\mathbb{R}^d \setminus \Gamma^T_\omega)$ is contained in the domain of H^T_ω and*

$$H^T_\omega f = -\Delta f, \qquad f \in C_0^\infty(\mathbb{R}^d \setminus \Gamma^T_\omega).$$

Then $\mathbb{W}$-a.s. the operator H^T_ω has no positive eigenvalue.

Proof. Let $\omega \in \Omega$ be such that Γ^T_ω has Lebesgue measure zero and its complement $\mathbb{R}^d \setminus \Gamma^T_\omega$ is connected. Then the set $C_0^\infty(\mathbb{R}^d \setminus \Gamma^T_\omega)$ is dense in $L^2(\mathbb{R}^d, \lambda^d)$ and the adjoint of the restriction of $-\Delta$ to this space is an extension of H^T_ω,

$$H^T_\omega \subset \left(-\Delta \restriction C_0^\infty(\mathbb{R}^d \setminus \Gamma^T_\omega)\right)^* =: -\Delta^T_{\omega,\max}.$$

Let $E > 0$ and $H^T_\omega f = Ef$. Then

$$\int_{\mathbb{R}^d} E\bar{f}(x)g(x)\lambda^d(\mathrm{d}x) = \left(-\Delta^T_{\omega,\max} f, g\right)$$
$$= \int_{\mathbb{R}^d} \bar{f}(x)(-\Delta g)(x)\lambda^d(\mathrm{d}x), \qquad g \in C_0^\infty(\mathbb{R}^d \setminus \Gamma^T_\omega).$$

By Weyl's regularity theorem, it follows that f is infinitely differentiable on $\mathbb{R}^d \setminus \Gamma^T_\omega$ and

$$H^T_\omega f(x) = -\sum_{j=1}^d \frac{\partial^2 f}{\partial x_j^2} = Ef \qquad \lambda^d\text{-a.e. on } \mathbb{R}^d \setminus \Gamma^T_\omega. \tag{4.16}$$

Let B be any ball containing Γ^T_ω. Since $-\sum_{j=1}^d \frac{\partial^2}{\partial x_j^2} f = Ef \lambda^d$-a.e. on the complement of B and $f \in L^2(\mathbb{R}^d, \lambda^d)$ we have $f = 0$ λ^d-a.e. on $\mathbb{R}^d \setminus B$ (cf, e.g., the proof of [RS4, theorem XIII.56]). By [RS4, theorem XIII.63] and (4.16), it follows that $f = 0\lambda^d$-a.e. on the connection component of $\mathbb{R}^d \setminus \Gamma^T_\omega$ containing B. Since $\mathbb{R}^d \setminus \Gamma^T_\omega$ is connected and the Lebesgue measure of the compact set Γ^T_ω equals zero it follows that $f = 0\lambda^d$-a.e. Thus E is not an eigenvalue of H^T_ω.

Since $d \geqslant 3$ and the two-dimensional Hausdorff measure of Γ^T_ω equals zero for $\mathbb{W}$-a.a. $\omega \in \Omega$ the Lebesgue measure of Γ^T_ω equals zero and the complement of Γ^T_ω is connected for $\mathbb{W}$-a.a. $\omega \in \Omega$. □

5. A trace formula for the expectation value of the number of negative eigenvalues

In this section, we shall derive a trace formula for the number of negative eigenvalues of the operators $H^T_{\omega\alpha}$ provided $3 \leqslant d \leqslant 5$. By mimicking the reasoning below and using the Klaus–Newton method (cf [Kl, Newt] and the extension in [BEKS]), similar results can be derived for $d = 1, 2$ as well.

Let

$$A_{\alpha 0} := \frac{(-\Delta + \alpha)(-\Delta)}{\alpha}. \tag{5.17}$$

By lemma 3.3 and (4.9), for $\mathbb{W}$-a.a. $\omega \in \Omega$ the operator J^T_ω from $H^2(\mathbb{R}^d)$ to $L^2\big(\mathbb{R}^d, \mu^T_\omega\big)$ (cf (2.10)) is compact and for every $\varepsilon > 0$ there exists an $a(\alpha, \varepsilon, \omega) < \infty$ such that

$$\big\| J^T_\omega f \big\|^2_{L^2(\mathbb{R}^d, \mu^T_\omega)} \leqslant \varepsilon \big\| A^{1/2}_{\alpha 0} f \big\|^2_{L^2(\mathbb{R}^d, \lambda^d)} + a(\alpha, \varepsilon, \omega) \| f \|^2_{L^2(\mathbb{R}^d, \lambda^d)}, \qquad f \in H^2(\mathbb{R}^d). \tag{5.18}$$

Thus $\mathbb{W}$-a.s. the quadratic form $\mathcal{E}^T_{\alpha 0\omega}$ in $L^2(\mathbb{R}^d, \lambda^d)$, defined by

$$D\big(\mathcal{E}^T_{\alpha 0\omega}\big) = H^2(\mathbb{R}^d), \tag{5.19}$$

$$\mathcal{E}^T_{\alpha 0\omega}(f, g) = \big(A^{1/2}_{\alpha 0} f, A^{1/2}_{\alpha 0} g\big) - \int \overline{\tilde{f}} \tilde{g} \, \mathrm{d}\mu^T_\omega, \qquad f, g \in H^2(\mathbb{R}^d), \tag{5.20}$$

is lower semibounded and closed. We denote by $A_{\alpha 0} - \mu^T_\omega$ the unique lower semibounded self-adjoint operator in $L^2(\mathbb{R}^d, \lambda^d)$ associated with $\mathcal{E}^T_{\alpha 0\omega}$.

Let $N_1(\omega, T)$ and $N_2(\omega, T)$ be the number (counting multiplicities) of negative eigenvalues of the operator $H^T_{\omega\alpha}$ and $A_{\alpha 0} - \mu^T_\omega$, respectively. By [Bra], corollary 8,

$$N_1(\cdot, T) = N_2(\cdot, T) \qquad \mathbb{W}\text{-a.s.} \tag{5.21}$$

Let

$$G_{\alpha 0\gamma} := (A_{\alpha 0} + \gamma)^{-1}. \tag{5.22}$$

By [Bra], (28),

$$\big(A_{\alpha 0} - \mu^T_\omega + \gamma\big)^{-1} = G_{\alpha 0\gamma} + \big(J^T_\omega G_{\alpha 0\gamma}\big)^* \big(1 - J^T_\omega \big(J^T_\omega G_{\alpha 0\gamma}\big)^*\big)^{-1} J^T_\omega G_{\alpha 0\gamma} \tag{5.23}$$

for every $\gamma > 0$ such that $-\gamma$ belongs to the resolvent set of $A_{\alpha 0} - \mu^T_\omega$. Let

$$K^T_{\omega\alpha\gamma} := 1_{(1,\infty)}\big(J^T_\omega \big(J^T_\omega G_{\alpha 0\gamma}\big)^*\big). \tag{5.24}$$

Modifying the Birman–Schwinger analysis in an obvious way, we can derive from (5.23) that the number of eigenvalues below $-\gamma$ of $A_{\alpha 0} - \mu^T_\omega$ equals $\big\| K^T_{\cdot\alpha\gamma} \big\|_{\mathbb{S}_1}$ $\mathbb{W}$-a.s.

In particular, the number of negative eigenvalues of $H^T_{\omega\alpha}$ is less than or equal to $\| G_{\alpha 00} \|^4_{\mathbb{S}_4}$. By the considerations in the proof of lemma 3.3 (cf, in particular, formula (3.8)), the expectation value of the last expression is finite if $3 \leqslant d \leqslant 5$. Thus we have proved the following theorem.

Theorem 5.6. *Let $3 \leqslant d \leqslant 5$ and $\alpha > 0$. For $\mathbb{W}$-a.a. $\omega \in \Omega$ let $H^T_{\omega\alpha}$ be the self-adjoint operator defined by (2.7), (2.10) and (2.11). Then for $\mathbb{W}$-a.a. $\omega \in \Omega$ the number (counting multiplicities) of negative eigenvalues of $H^T_{\omega\alpha}$ equals the trace norm of the operator $K^T_{\omega\alpha 0}$, defined by (5.24). In particular, the expectation value (w.r.t. Wiener measure) for the number (counting multiplicities) of negative eigenvalues of $H^T_{\omega\alpha}$ is finite.*

Remark 5.7. In a forthcoming paper, we shall derive further results on the negative eigenvalues. In particular, we shall show that for every $N \in \mathbb{N}$ the probability that the number of negative eigenvalues of $H^T_{\omega\alpha}$ is at least N is strictly positive. On the other hand, these probabilities tend rapidly to zero, as N tends to infinity. In fact, the above theorem implies that the sum over N times the probability that the number of negative eigenvalues of $H^T_{\omega\alpha}$ equals N is finite; here the sum is taken over all positive integers N.

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